

Central Limit Theorem for probability measures defined by sum-of-digits function in base 2

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Abstract

In this paper we prove a central limit theorem for some probability measures defined as asymptotic densities of integer sets defined via sum-of-digit-function. To any integer a we can associate a measure on \mathbb{Z} called μ_a such that, for any d , $\mu_a(d)$ is the asymptotic density of the set of integers n such that $s_2(n+a) - s_2(n) = d$ where $s_2(n)$ is the number of digits “1” in the binary expansion of n . We express this probability measure as a product of matrices. Then we take a sequence of integers $(a_X(n))_{n \in \mathbb{N}}$ via a balanced Bernoulli process. We prove that, for almost every sequence, and after renormalization by the typical variance, we have a central limit theorem by computing all the moments and proving that they converge towards the moments of the normal law $\mathcal{N}(0, 1)$.

1 Introduction

1.1 Background

In this paper we study some properties of sets defined via sum-of-digit function in base 2. Namely, for a given integer a , we study the asymptotic density of set of integers such that the difference of 1 in their binary expansion before and after addition with a is a given integer. More precisely, we define:

$$\forall n \in \mathbb{N}, s_2(n) = \sum_{k=0}^m n_k$$

where

$$n = \sum_{k=0}^m n_k 2^k,$$

and

$$\forall a \in \mathbb{N}, \forall d \in \mathbb{Z}, \mu_a(d) = \lim_{N \rightarrow +\infty} \frac{1}{N} \#\{n < N \mid s_2(n+a) - s_2(n) = d\}.$$

This can be linked with correlation functions that are studied in [1] for instance, or with properties of sum-of-digit functions have been extensively studied in, for instance [5] or, more recently, in [8]. We can also quote [10] for the links between THUE-MORSE sequence and the sum-of-digits function in base 2.

More precisely, we are interested in normality properties of such sets and we give in this paper a central limit theorem for a random.. This kind of properties have raised a considerable interest in number theory. We can quote [4, 6, 11] for some of these normality properties for q -additive functions.

In [9], we were interested in those densities of sets and more precisely in their asymptotic properties as a grew bigger. The methods for computing those densities were essentially combinatorial. In this paper, we are closer to dynamical systems as we study a random product of matrices.

1.2 Results

Definition 1.1. Let $n \in \mathbb{N}$. There exists a unique smallest $m \in \mathbb{N}$ and a unique sequence $\{n_0, \dots, n_m\} \in \{0, 1\}^m$ such that:

$$n = \sum_{k=0}^m n_k \cdot 2^k.$$

Denote $\underline{n}_2 = n_m \dots n_0$.

Definition 1.2. Define the sum-of-digits function in base 2 s_2 , as:

$$\begin{aligned} s_2 : \mathbb{N} &\rightarrow \mathbb{N} \\ n &\mapsto \sum_{k=0}^m n_k \end{aligned}$$

where $\underline{n}_2 = n_m \dots n_0$.

We are interested in the following equation with parameters $a \in \mathbb{N}$, $d \in \mathbb{Z}$ and unknown $n \in \mathbb{N}$:

$$s_2(n + a) - s_2(n) = d.$$

More precisely, we wish to understand the asymptotic densities of the following sets:

$$\mathcal{E}_{a,d} := \{n \in \mathbb{N} \mid s_2(n + a) - s_2(n) = d\}.$$

In [9], we prove the following:

Proposition 1.3. For any $a \in \mathbb{N}$, $d \in \mathbb{Z}$, there exists a finite set of words $\mathcal{P}_{a,d} := \{w_1, \dots, w_k\} \subset \{0, 1\}^*$ such that:

$$\mathcal{E}_{a,d} = \bigcup_{i \in \{1, \dots, k\}} [w_i]$$

where $[w]$ is the set of integers n such that \underline{n}_2 ends with w .

Remark 1.4. Remark that $\mathcal{P}_{a,d}$ is finite and, possibly, empty (whenever $d > s_2(a)$ actually).

From Proposition 1.3, it is clear that the densities of the sets $\mathcal{E}_{a,d}$ exist. This was known since [1]. In this way we define the main object of our study:

Definition 1.5. Let us define, for any $a \in \mathbb{N}$, the probability measure μ_a by:

$$\forall d \in \mathbb{Z}, \mu_a(d) := \lim_{N \rightarrow +\infty} \frac{\#\{n \leq N \mid s_2(n + a) - s_2(n) = d\}}{N}.$$

We are interested in asymptotic properties of the measures μ_a as a goes to infinity in a certain sense. Namely, we define the following quantity:

Definition 1.6. Let $a \in \mathbb{N}$. define the quantity:

$$l(a) := \#\{\text{occurences of "01" in } \underline{a}_2\}$$

We recall two theorems from [9]:

Theorem 1.7. *There exists a constant $C > 0$ such that, for any $a \in \mathbb{N}$:*

$$\|\mu_a\|_2 \leq C \cdot l(a)^{-1/4}.$$

Theorem 1.8. *For any $a \in \mathbb{N}$, the probability measure μ_a has mean 0 and its variance is bounded by:*

$$l(a) - 1 \leq \text{Var}(\mu_a) \leq 4l(a) + 2.$$

This last theorem raises the question of finding an equivalence for the variance of the probability measure μ_a . We do this in the generic case for the balanced Bernoulli measure. More precisely, we have the following central limit theorem:

Theorem 1.9. *Let $X = (X_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ be a generic sequence for the balanced Bernoulli measure. Define the sequence $(a_X(n))_{n \in \mathbb{N}}$ in the following way:*

$$a_X(n) = \sum_{k=0}^n X_k \cdot 2^k.$$

For any $n \in \mathbb{N}$, let $\tilde{\mu}_{a_X(n)} \in l^1\left(\sqrt{\frac{2}{n}}\mathbb{Z}\right)$ defined by

$$\forall d \in \sqrt{\frac{2}{n}}\mathbb{Z}, \quad \tilde{\mu}_{a_X(n)} = \mu_{a_X(n)}\left(\sqrt{\frac{n}{2}}d\right).$$

Then

$$\mu_{a_X(n)} \xrightarrow[n \rightarrow +\infty]{\text{weak}} \varphi$$

where

$$\begin{aligned} \varphi : \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \end{aligned}$$

Remark 1.10. An equivalent formulation, with the same notations, is:

$$\forall x \in \mathbb{R}, \quad \lim_{n \rightarrow +\infty} \lim_{N \rightarrow +\infty} \frac{1}{N} \# \left\{ m \leq N \mid \frac{s_2(m+a) - s_2(m)}{\sqrt{\frac{n}{2}}} \leq x \right\} = \Phi(x)$$

where Φ is the repartition function of the normal law $\mathcal{N}(0, 1)$.

Finally we wish to state CUSICK's conjecture: Let $s_2(n)$ denote, for any integer n , the number of 1 in the binary expansion of n and define:

$$\forall a \in \mathbb{N}, \quad \gamma_a := \lim_{N \rightarrow +\infty} \frac{1}{N} \# \{n \leq N \mid s_2(n+a) \geq s_2(n)\}.$$

The conjecture consists of two parts:

$$\forall a \in \mathbb{N}, \gamma_a \geq \frac{1}{2}$$

and

$$\liminf_{a \rightarrow +\infty} \gamma_a = \frac{1}{2}.$$

This question arose as CUSICK was working on a similar combinatorial conjecture in [3]. For recent advances on this, one can look at [13] and [7]. In particular, the main theorem of this paper has for immediate corollary that $\frac{1}{2}$ is an accumulation point for the sequence $(\gamma_a)_{a \in \mathbb{N}}$.

Moreover, the proof of our theorem answers a question left open in [7] since it states that the difference $s_2(n+a) - s_2(n)$ is “usually normally distributed with mean zero and variance $\frac{|a|_2}{2}$ ” where $|a|_2$ denotes the length of a in base 2.

1.3 Outline of the paper

The goal of this paper is to demonstrate Theorem 1.9 by a moments method. Namely, given a sequence of probability measures, we prove the weak convergence towards the normal law $\mathcal{N}(0, 1)$ by proving that all the moments of this sequence converge towards the moments of the normal law.

This article is organised as follows:

Section 2 deals with the measures μ_a that we already studied in [9]. We recall some of their properties (and most importantly a recurrence relation between them) and write them as a finite product of matrices whose coefficients are operators on $l^1(\mathbb{Z})$. This is a convenient form for our study since it allows to compute the Fourier transforms of these measures explicitly.

Section 3 is devoted to the proof of Theorem 1.9. It is divided in the following way:

In Subsection 3.1, we explicit this Fourier transform and give its Taylor expansion around 0 at order 2. This is what we need in order to compute the variance of μ_a . We also mention that the characteristic function can be written as a product of matrices, which is the form that will be studied throughout the article.

Subsection 3.2 is devoted to the computation of the variance of μ_a . First we do this in the general case and give an explicit formula depending only on the binary decomposition of a . We remark that this expression depends on some correlations of sequences in $\{0, 1\}^{\mathbb{N}}$.

Then, in Subsection 3.3, we want to compute the “generic” behaviour of μ_a (meaning for a a whose binary expansion is given by a balanced Bernoulli sequence). For this, we use a result in [2] to estimate the correlation terms. It appears that in the generic case, the variance is approximately $\frac{|a|_2}{2}$ (where $|a|_2$ is the length of a_2). So we know that in order to get a central limit theorem, we have to renormalize μ_a by the squareroot of its variance namely $\sqrt{\frac{|a|_2}{2}}$.

In Subsection 3.4, since we have to compute all the moments, we need to know all the coefficients in the Taylor expansion of the characteristic function but in the general case this proves to be too hard a task. Hence we wish to understand how “big” the different terms are in order to know which one will be killed by the renormalization and which one will contribute.

Finally, in Subsection 3.5, we show that the moments converge towards the moments of the normal law. Thanks to the study of the contributions from the previous section, we are limited to actually computing the terms that have a chance to contribute. Some elementary linear algebra and the study of the correlations appearing in Section 3.2 are the essential tools for this.

1.4 Acknowledgement

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2 Measures μ_a on \mathbb{Z}

Let us start by remarking the following:

Remark 2.1. Let $a \in \mathbb{N}$ and $d \in \mathbb{Z}$.

$$\mathcal{P}_{2a,d} = \{w0 \mid w \in \mathcal{P}_{a,d}\} \cup \{w1 \mid w \in \mathcal{P}_{a,d}\}$$

and

$$\mathcal{P}_{2a+1,d} = \{w0 \mid w \in \mathcal{P}_{a,d-1}\} \cup \{w1 \mid w \in \mathcal{P}_{a+1,d+1}\}.$$

For more details about this remark we refer the reader to [9].

From this we deduce the following:

Proposition 2.2. *For any $a \in \mathbb{N}$:*

$$\mu_{2a} = \mu_a$$

and

$$\mu_{2a+1}(d) = \frac{1}{2}\mu_a(d-1) + \frac{1}{2}\mu_{a+1}(d+1).$$

Remark 2.3. Notice that a probability measure on \mathbb{Z} is, in particular, an element of $l^1(\mathbb{Z})$. In all that follows, for simplicity of writing, we will always identify a measure on \mathbb{Z} with its associated sequence in $l^1(\mathbb{Z})$. In particular, we see μ_a both as a measure and as an element of $l^1(\mathbb{Z})$. Let us define the shift S on $l^1(\mathbb{Z})$.

$$\begin{aligned} S : \quad l^1(\mathbb{Z}) &\rightarrow l^1(\mathbb{Z}) \\ (x_n)_{n \in \mathbb{Z}} &\mapsto (x_{n+1})_{n \in \mathbb{Z}}. \end{aligned}$$

Then, the identities of Proposition 2.2 can be written:

$$\mu_{2a} = \mu_a$$

and

$$\mu_{2a+1} = \frac{1}{2}S^{-1}(\mu_a) + \frac{1}{2}S(\mu_a)$$

Example 2.4. It is easy to see that $\mu_0 = \delta_0$. Then, either by standard computation or using Proposition 2.2, one obtains:

$$\mu_1 = \frac{1}{4} \sum_{\substack{n \in \mathbb{Z} \\ n \leq 1}} \delta_n \cdot 2^n.$$

Proposition 2.5. *For any $a \in \mathbb{N}$,*

$$\mu_a = \begin{pmatrix} Id & 0 \end{pmatrix} A_{a_0} \cdots A_{a_{n-1}} A_{a_n} \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix},$$

where the sequence $\underline{a}_2 = a_n \dots a_0$ and

$$A_0 = \begin{pmatrix} Id & 0 \\ \frac{1}{2}S^{-1} & \frac{1}{2}S \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{1}{2}S^{-1} & \frac{1}{2}S \\ 0 & Id \end{pmatrix},$$

Proof. Suffice to remark that for any $a \in \mathbb{N}$,

$$A_0 \begin{pmatrix} \mu_a \\ \mu_{a+1} \end{pmatrix} = \begin{pmatrix} \mu_{2a} \\ \mu_{2a+1} \end{pmatrix}$$

and

$$A_1 \begin{pmatrix} \mu_a \\ \mu_{a+1} \end{pmatrix} = \begin{pmatrix} \mu_{2a+1} \\ \mu_{2a+2} \end{pmatrix}$$

with Proposition 2.2. □

Notice that this proposition is a much clearer version of Theorem 1.2.1 in [9]

3 Central limit theorem

3.1 Characteristic function

Let $a \in \mathbb{N}$ with $\underline{a}_2 = a_n \dots a_0$. The characteristic function of μ_a , noted $\widehat{\mu}_a$ is defined in the standard way:

$$\forall \theta \in [0, 2\pi), \widehat{\mu}_a(\theta) = \sum_{d \in \mathbb{Z}} e^{id\theta} \mu_a(d)$$

where

$$\forall \theta \in [0, 2\pi), \hat{A}_0(\theta) := \begin{pmatrix} 1 & 0 \\ \frac{1}{2}e^{i\theta} & \frac{1}{2}e^{-i\theta} \end{pmatrix}, \hat{A}_1(\theta) := \begin{pmatrix} \frac{1}{2}e^{i\theta} & \frac{1}{2}e^{-i\theta} \\ 0 & 1 \end{pmatrix}.$$

By Proposition 2.5, the characteristic function $\widehat{\mu}_a : [0, 2\pi) \rightarrow \mathbb{C}$ given by:

$$\forall \theta \in [0, 2\pi), \widehat{\mu}_a(\theta) = \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{A}_{a_0} \cdots \hat{A}_{a_{n-1}} \hat{A}_{a_n} \begin{pmatrix} \widehat{\mu}_0(\theta) \\ \widehat{\mu}_1(\theta) \end{pmatrix}.$$

A quick computation yields

$$\forall \theta \in [0, 2\pi), \widehat{\mu}_0(\theta) = 1, \quad \widehat{\mu}_1(\theta) = \frac{e^{i\theta}}{2 - e^{-i\theta}}$$

and so,

$$\forall \theta \in [0, 2\pi), \widehat{\mu}_a(\theta) = \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{A}_{a_0} \cdots \hat{A}_{a_{n-1}} \hat{A}_{a_n} \begin{pmatrix} 1 \\ \frac{e^{i\theta}}{2 - e^{-i\theta}} \end{pmatrix}.$$

Let us now define the matrices playing a role in the Taylor expansion of $\widehat{\mu}_a$ near 0.

$$I_0 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \alpha_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad \beta_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$I_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad \alpha_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad \beta_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Indeed, we have:

$$\hat{A}_j(\theta) = I_j + i\theta\alpha_j - \frac{1}{2}\theta^2\beta_j + O(\theta^3).$$

with $j \in \{0, 1\}$.

And notice that the Taylor expansion near 0 of $\theta \mapsto \frac{e^{-i\theta}}{2 - e^{i\theta}}$ is:

$$\frac{e^{i\theta}}{2 - e^{-i\theta}} = 1 - \theta^2 + O(\theta^3).$$

3.2 Computation of the variance

Define the variance of μ_a :

$$\text{Var}(\mu_a) = \sum_{d \in \mathbb{Z}} \mu_a(d) d^2.$$

Theorem 3.1. For any $a \in \mathbb{N}$ with $\underline{a}_2 = a_n \dots a_0$, denote, for any $j \in \{0, \dots, n\}$, $b_j = (-1)^{a_j+1}$. The variance of μ_a is given by the following:

$$\text{Var}(\mu_a) = \frac{n+3}{2} - \frac{1}{2^{n+1}} + b_0 - \frac{1}{2} \sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}} + \sum_{k=0}^{n-1} \frac{b_{k+1}}{2^{k+2}} + \sum_{i=0}^n \frac{b_i}{2^{n+1-i}}.$$

Proof. Notice that the variance is given by:

$$\text{Var}(\mu_a) = \begin{pmatrix} 1 & 0 \end{pmatrix} (\beta_{a_0} + I_{a_0}\beta_{a_1} + \dots + I_{a_0} \dots I_{a_{n-1}}\beta_{a_n}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} I_{a_0} \dots I_{a_n} \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Since $\alpha_j \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$ and $I_j \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We now apply a change of basis to simultaneously trigonalise the matrices I_0 and I_1 .

Let us note

$$P := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and compute

$$\forall j \in \{0, 1\}, \quad \tilde{I}_j := P I_j P^{-1} = \begin{pmatrix} 1 & \frac{(-1)^{j+1}}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \tilde{\beta}_j := P \beta_j P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{(-1)^{j+1}}{2} & 0 \end{pmatrix}$$

and

$$P \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad P \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

With this change of basis, the variance becomes:

$$\text{Var}(\mu_a) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} (\tilde{\beta}_{a_0} + \tilde{I}_{a_0}\tilde{\beta}_{a_1} + \dots + \tilde{I}_{a_0} \dots \tilde{I}_{a_{n-1}}\tilde{\beta}_{a_n}) \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tilde{I}_{a_0} \dots \tilde{I}_{a_n} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Notice now that for any $k \in \{0, \dots, n\}$:

$$\tilde{I}_{a_0} \dots \tilde{I}_{a_k} = \begin{pmatrix} 1 & \sum_{i=0}^k \frac{b_i}{2^{k+1-i}} \\ 0 & \frac{1}{2^{k+1}} \end{pmatrix}$$

so, for any $k \in \{0, \dots, n-1\}$:

$$\tilde{I}_{a_0} \dots \tilde{I}_{a_k} \tilde{\beta}_{a_{k+1}} = \begin{pmatrix} \frac{1}{2} - \frac{b_{k+1}}{2} \sum_{i=0}^k \frac{b_i}{2^{k+1-i}} & 0 \\ -\frac{b_{k+1}}{2^{k+2}} & 0 \end{pmatrix}$$

hence we get

$$\begin{aligned} \text{Var}(\mu_a) = & \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{b_0}{2} & 0 \end{pmatrix} + \sum_{k=0}^{n-1} \begin{pmatrix} \frac{1}{2} - \frac{b_{k+1}}{2} \sum_{i=0}^k \frac{b_i}{2^{k+1-i}} & 0 \\ k - \frac{b_{k+1}}{2^{k+2}} & 0 \end{pmatrix} \right) \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & \sum_{i=0}^n \frac{b_i}{2^{n+1-i}} \\ 0 & \frac{1}{2^{n+1}} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \end{aligned}$$

so

$$\text{Var}(\mu_a) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ -b_0 \end{pmatrix} + \sum_{k=0}^{n-1} \begin{pmatrix} 1 - b_{k+1} \sum_{i=0}^k \frac{b_i}{2^{k+1-i}} \\ k - \frac{b_{k+1}}{2^{k+2}} \end{pmatrix} \right) + 1 + \sum_{i=0}^n \frac{b_i}{2^{n+1-i}} - \frac{1}{2^{n+1}}$$

which yields the result:

$$\text{Var}(\mu_a) = \frac{n+3}{2} - \frac{1}{2^{n+1}} + b_0 - \frac{1}{2} \sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}} + \sum_{k=0}^{n-1} \frac{b_{k+1}}{2^{k+2}} + \sum_{i=0}^n \frac{b_i}{2^{n+1-i}}.$$

□

3.3 Generic case of the variance

In all that follows, we use the following notations:

- X denotes a sequence in $\{0, 1\}^{\mathbb{N}}$ (and we endow the set $\{0, 1\}^{\mathbb{N}}$ with the balanced Bernoulli probability measure.
- For any sequence X , define $a_X(n) = \sum_{k=0}^n X_k \cdot 2^k$
- As in Theorem 3.1, for any $X \in \{0, 1\}^{\mathbb{N}}$, define the sequence $(b_j)_{j \in \mathbb{N}}$ by $b_j = (-1)^{X_{j+1}}$.

We wish to prove the following:

Proposition 3.2. *For almost every $X \in \{0, 1\}^{\mathbb{N}}$*

$$\text{Var}(\mu_{a_X(n)}) \underset{n \rightarrow \infty}{\sim} \frac{n}{2}.$$

In order to prove this proposition, we first need a technical lemma:

Lemma 3.3. *Let $X \in \{0, 1\}^{\mathbb{N}}$ and define the quantity:*

$$C_{2,n} = \max_{M,D} \left| \sum_{k=1}^M b_{k+d_1} b_{k+d_2} \right|,$$

where the maximum is taken on all $D = (d_1, d_2)$ and M such that $M + d_2 \leq n$.

For almost every $X \in \{0, 1\}^{\mathbb{N}}$ and for every ε there exists $n_{\varepsilon, X}$ such that:

$$\forall n \geq n_{\varepsilon, X}, \quad |C_{2,n}| < n^{\frac{1}{2} + \varepsilon}$$

Proof. In [2], the following quantity is studied:

$$C_m((b_i)_{i \in \{0, \dots, n\}}) := \max_{M,D} \left| \sum_{k=1}^M b_{k+d_1} \dots + b_{k+d_m} \right|,$$

where the maximum is taken on all $D = (d_1, \dots, d_m)$ and M such that $M + d_m \leq n$.

So we have:

$$C_{2,n} = C_2((b_i)_{i \in \{0, \dots, n\}}).$$

From (2.32) and (2.33) in [2], we know that, for any $l \geq 1$:

$$\mathbb{E} \left(C_2((b_i)_{i \in \{0, \dots, n\}})^{2l} \right) \leq 5n^{4+l} (4l)^l.$$

Let $\varepsilon > 0$,

$$\mathbb{E} \left(\frac{C_{2,n}^{2l}}{n^{(\frac{1}{2}+\varepsilon)2l}} \right) \leq \frac{5n^{4+l}(4l)^l}{n^{(\frac{1}{2}+\varepsilon)2l}}.$$

and

$$\frac{5n^{4+l}(4l)^l}{n^{(\frac{1}{2}+\varepsilon)2l}} = \frac{5(4l)^l}{n^{2\varepsilon l-4}}.$$

Now, if l is big enough, then $2\varepsilon l - 4 > 2$ and thus the series

$$\sum_{n=1}^{+\infty} \mathbb{E} \left(\frac{C_{2,n}^{2l}}{n^{(\frac{1}{2}+\varepsilon)2l}} \right)$$

converges.

By Borel-Cantelli lemma, $\frac{C_{2,n}^{2l}}{n^{(\frac{1}{2}+\varepsilon)2l}} \xrightarrow[n \rightarrow +\infty]{a.s.} 0$ and thus $\frac{C_{2,n}}{n^{(\frac{1}{2}+\varepsilon)}} \xrightarrow[n \rightarrow +\infty]{a.s.} 0$. Hence, for almost every X , $\exists n_{\varepsilon, X}$, $\forall n \geq n_{\varepsilon, X}$,

$$\frac{C_{2,n}}{n^{(\frac{1}{2}+\varepsilon)}} < 1$$

and thus

$$|C_{n,j}| < n^{\frac{1}{2}+\varepsilon}$$

for n big enough. □

Let us now prove Proposition 3.2.

Proof of Proposition 3.2. Note that, with Theorem 3.1, for any n ,

$$\text{Var}(\mu_{a_X(n)}) = \frac{n+3}{2} - \frac{1}{2^{n+1}} + b_0 - \frac{1}{2} \sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}} + \sum_{k=0}^{n-1} \frac{b_{k+1}}{2^{k+2}} + \sum_{i=0}^n \frac{b_i}{2^{n+1-i}}.$$

where the b_i are random variables which can take value in $\{-1, 1\}$ with probability $\frac{1}{2}$. The only thing to prove in order to get the result is that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}} \right) = 0$$

since it is obvious that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{3}{2} - \frac{1}{2^{n+1}} + b_0 - \sum_{k=0}^{n-1} \frac{b_{k+1}}{2^{k+2}} + \sum_{i=0}^n \frac{b_i}{2^{n+1-i}} \right) = 0.$$

Let us estimate $\sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}}$.

$$\begin{aligned} \sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1}b_i}{2^{k+1-i}} &= \sum_{k=0}^{n-1} \sum_{i=0}^k \frac{b_{k+1}b_i}{2^{k+1-i}} \\ &= \sum_{j=1}^n \frac{1}{2^j} \sum_{k=j}^n b_k b_{k-j}. \end{aligned}$$

Hence, with Lemma 3.3, for any $\varepsilon > 0$, for any n big enough:

$$\begin{aligned} \left| \sum_{0 \leq i \leq k \leq n-1} \frac{b_{k+1} b_i}{2^{k+1-i}} \right| &\leq \sum_{j=1}^n \frac{|C_{2,n}|}{2^j} \\ &\leq \sum_{j=1}^n \frac{n^{\frac{1}{2}+\varepsilon}}{2^j} \\ &\leq n^{\frac{1}{2}+\varepsilon}, \end{aligned}$$

which ends the proof. □

3.4 Upper bounds of the moments of $\mu_{a_X(n)}$

We now want to have bound on the moments of order $l \in \mathbb{N}$. Let us first remark that the only matrices appearing in the Taylor expansion of $\hat{A}_i(\theta)$ are I_i , α_i and β_i . Indeed:

$$\hat{A}_i(\theta) = \sum_{j=0}^{+\infty} \theta^j T_{i,j},$$

where

$$T_{i,0} = I_i, \quad T_{i,2j} = \frac{(-1)^j}{(2j)!} \beta_i, \quad T_{i,2j+1} = \frac{(-1)^j i}{(2j+1)!} \alpha_i$$

Remark 3.4. Notice that the following relations hold:

$$I_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = I_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \alpha_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$I_0 \alpha_0 = I_1 \alpha_0 = \frac{1}{2} \alpha_0, \quad I_0 \alpha_1 = I_1 \alpha_1 = \frac{1}{2} \alpha_1.$$

Let us insist on the fact that these relations are crucial for our proof.

Let us now introduce the following norm on 2×2 matrices:

$$\|M\| = \max_{i \in \{1,2\}} (|M_{i,1}| + |M_{i,2}|)$$

which is induced by $\|\cdot\|_1$ on \mathbb{R}^2 .

Notice that this defines a submultiplicative norm. Moreover,

$$\|I_0\| = \|I_1\| = \|\alpha_0\| = \|\alpha_1\| = \|\beta_0\| = \|\beta_1\| = 1.$$

Our goal is to compute all the moments of the probability measure $\mu_{a_X(n)}$ in the generic case as n goes to infinity. To that end, we arrange the terms appearing in the computation into different “types”. A type is a couple (α^p, β^q) where p, q are non negative integers. They indicate the number of matrices of α and β appearing in the term. For instance, a term:

$$M = I_{a_0} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}} I_{a_{i_0+1}} \cdots I_{a_{i_1-1}} \alpha_{a_{i_1}} I_{a_{i_1+1}} \cdots I_{a_{i_2-1}} \beta_{a_{i_2}} I_{a_{i_2+1}} \cdots I_{a_n}$$

is of type (α^2, β^1) . The order of appearance of the α and β does not have any influence on the type, so that

$$I_{a_0} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}} I_{a_{i_0+1}} \cdots I_{a_{i_1-1}} \beta_{a_{i_1}} I_{a_{i_1+1}} \cdots I_{a_{i_2-1}} \alpha_{a_{i_2}} I_{a_{i_2+1}} \cdots I_{a_n}$$

is also of type (α^2, β^1) .

Let us denote by $\mathcal{F}_{(\alpha^p, \beta^q)}^{(n)}$ the set of all terms of type (α^p, β^q) in the expansion of $\mu_{a_X(n)}$.

This notation is introduced to ease the writing of the previous formulas as well as for handling terms with the same behaviour together. For instance, the formula for the variance becomes:

$$\text{Var}(\mu_{a_X(n)}) = (1 \ 0) \sum_{M \in \mathcal{F}_{(\alpha^2, \beta^0)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1 \ 0) \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^1)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1 \ 0) I_{a_0} \cdots I_{a_n} \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Now notice that for any 2×2 matrix M ,

$$\left| (1 \ 0) M \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right| \leq \|M\|$$

so in order to estimate terms of a given type, suffices to understand:

$$\sum_{M \in \mathcal{F}_{(\alpha^p, \beta^q)}^{(n)}} \|M\|.$$

Definition 3.5. We say that a type (α^p, β^q) contributes with weight at most k if:

$$\sum_{M \in \mathcal{F}_{(\alpha^p, \beta^q)}^{(n)}} \|M\| = O(n^k)$$

Lemma 3.6. For any pair of nonnegative integers (p, q) , the type (α^p, β^q) contributes with weight at most q .

Proof. We prove this lemma by induction on p .

First notice that $\#\mathcal{F}_{(\alpha^0, \beta^q)}^{(n)} = \binom{n+1}{q}$. Notice also that for any $M \in \mathcal{F}_{(\alpha^0, \beta^q)}^{(n)}$, $\|M\| \leq 1$ since $\|\cdot\|$ is submultiplicative. This implies that the type (α^0, β^q) contributes with weight q .

Now let us assume that the type (α^p, β^q) contributes with weight q for a given p and let us prove that the type (α^{p+1}, β^q) has same weight. Now let us partition $\mathcal{F}_{(\alpha^{p+1}, \beta^q)}^{(n)}$. Fix $k \leq q$ and let us estimate terms that can be written $M \beta_{a_j} I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}} I_{a_{i_0+1}} \cdots I_{a_{i_1-1}} \beta_{a_{i_1}} \cdots \beta_{a_{i_k}}$ where $M \in \mathcal{F}_{(\alpha^p, \beta^{q-k-1})}^{(j-1)}$:

$$\begin{aligned}
& \sum_{j=p+q-k}^{n-k-1} \sum_{M \in \mathcal{F}_{(\alpha^p, \beta^{q-k-1})}^{(j-1)}} \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|M \beta_{a_j} I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}} I_{a_{i_0+1}} \cdots I_{a_{i_1-1}} \beta_{a_{i_1}} \cdots \beta_{a_{i_k}}\| \\
& \leq \sum_{j=p+q-k}^{n-k-1} \sum_{M \in \mathcal{F}_{(\alpha^p, \beta^{q-k-1})}^{(j-1)}} \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|M \beta_{a_j} I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \leq \sum_{j=p+q-k}^{n-k-1} \sum_{M \in \mathcal{F}_{(\alpha^p, \beta^{q-k-1})}^{(j-1)}} \|M\| \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \stackrel{\text{by induction}}{\leq} \sum_{j=p+q-k}^{n-k-1} C j^{q-k-1} \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \leq C n^{q-k-1} \sum_{j=p+q-k}^{n-k-1} \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \leq C n^{q-k} \sum_{j+1 \leq i_0 \leq \dots \leq i_k \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \leq C n^{q-k} \binom{n-j}{k} \sum_{j+1 \leq i_0 \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\|
\end{aligned}$$

and with Remark 3.4,

$$\begin{aligned}
& C n^{q-k} \binom{n-j}{k} \sum_{j+1 \leq i_0 \leq n} \|I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}}\| \\
& \leq C n^q \sum_{j+1 \leq i_0 \leq n} \frac{1}{2^{i_0-j}} \\
& \leq C n^q.
\end{aligned}$$

The same can be done for terms of the form $M \alpha_{a_j} I_{a_{j+1}} \cdots I_{a_{i_0-1}} \alpha_{a_{i_0}} I_{a_{i_0+1}} \cdots I_{a_{i_1-1}} \beta_{a_{i_1}} \cdots \beta_{a_{i_k}}$ where $M \in \mathcal{F}_{(\alpha^p, \beta^{q-k-1})}^{(j-1)}$. Hence:

$$\sum_{M \in \mathcal{F}_{(\alpha^{p+1}, \beta^q)}^{(n)}} \|M\| \leq 2q C n^q,$$

which proves the lemma □

3.5 Computing all the moments

Let us write the expansion of $\hat{\mu}_a$:

$$\hat{\mu}_a(\theta) = \sum_{k=0}^N \frac{i^k m_k(a)}{k!} \theta^k + o(\theta^n)$$

where

$$m_k(a) = \sum_{d \in \mathbb{Z}} \mu_a(d) d^k$$

is the moment of order k of the probability measure μ_a (indeed, recall that μ_a is centered).

Now let us renormalize $\mu_{a_X(n)}$. From Proposition 3.2, we know that we have to look at $\tilde{\mu}_{a_X(n)} \in l^1\left(\sqrt{\frac{2}{n}}\mathbb{Z}\right)$ defined by:

$$\forall d \in \sqrt{\frac{2}{n}}\mathbb{Z}, \quad \tilde{\mu}_{a_X(n)}(d) = \mu_{a_X(n)}\left(\sqrt{\frac{n}{2}}d\right).$$

Now notice that the characteristic function of $\tilde{\mu}_{a_X(n)}$ in θ is actually:

$$\hat{\mu}_{a_X(n)}\left(\sqrt{\frac{2}{n}}\theta\right) = \sum_{k=0}^N \frac{(i\sqrt{2})^k m_k(a_X(n))}{n^{\frac{k}{2}} k!} \theta^k + o(\theta^N).$$

Hence, for any $n \in \mathbb{N}$, the moments of order k of the probability measure $\tilde{\mu}_{a_X(n)}$, denoted by $\tilde{m}_k(a_X(n))$, are:

$$\tilde{m}_k(a_X(n)) = \frac{\sqrt{2^k} m_k(a_X(n))}{n^{\frac{k}{2}}}$$

and thus, we wish to understand, if it exists, for any integer k ,

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{2^k} m_k(a_X(n))}{n^{\frac{k}{2}}}.$$

Lemma 3.7. *For almost every sequence $X \in \{0, 1\}^{\mathbb{N}}$, for any $k \in \mathbb{N}$:*

$$\lim_{n \rightarrow +\infty} \tilde{m}_{2k}(a_X(n)) = \frac{(2k)!}{2^k k!}$$

and

$$\lim_{n \rightarrow +\infty} \tilde{m}_{2k+1}(a_X(n)) = 0$$

which are the moments of the normal law $\mathcal{N}(0, 1)$.

Proof. Let us remark right away that for a moment of order $2k+1$, the type of terms which could contribute the most is (α^1, β^k) . With Lemma 3.6, this type has weight at most k . Moreover, there is always a finite number of types contributing to a moment. Hence $m_{2k+1}(a_X(n)) = O(n^k)$ and thus:

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{2^{2k+1}} m_{2k+1}(a_X(n))}{n^{\frac{2k+1}{2}}} = 0.$$

or, equivalently,

$$\lim_{n \rightarrow +\infty} \tilde{m}_{2k+1}(a_X(n)) = 0.$$

Next, we consider the even moments.

For a moment m_{2k} , from Lemma 3.6, the only type potentially contributing to the limit is (α^0, β^k) . More precisely we have:

$$\lim_{n \rightarrow +\infty} \frac{(i\sqrt{2})^{2k} m_{2k}(a_X(n))}{n^k (2k)!} = \lim_{n \rightarrow +\infty} \left(\frac{2}{n}\right)^k \left(\frac{-1}{2}\right)^k \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

hence

$$\lim_{n \rightarrow +\infty} \frac{2^k m_{2k}(a_X(n))}{n^k} = \lim_{n \rightarrow +\infty} \frac{(2k)!}{n^k} \begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

In short, we must show that:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{n^k}{2^k k!} + o(n^k),$$

so that

$$\lim_{n \rightarrow +\infty} \frac{2^k m_{2k}(a_X(n))}{n^k} = \frac{(2k)!}{2^k k!}$$

which is the moment of order $2k$ of the normal law $\mathcal{N}(0, 1)$.

Let $\mathcal{D}_k(n) = \{(d_1, \dots, d_k) \mid 0 \leq d_1 < \dots < d_k \leq n\}$. For $d \in \mathcal{D}_k(n)$, denote $\Pi_d = \tilde{I}_{a_0} \cdots \tilde{I}_{a_{d_1-1}} \tilde{\beta}_{a_{d_1}} \cdots \tilde{I}_{a_{d_k-1}} \tilde{\beta}_{a_{d_k}}$ (this is just a matrix $M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}$ after the change of basis described in the proof of Theorem 3.1). Let us prove by induction on k that:

$$\Pi_d = \begin{pmatrix} \frac{1}{2^k} + A_d & 0 \\ B_d & 0 \end{pmatrix}$$

with A_d and B_d satisfying:

$$\lim_{n \rightarrow +\infty} \frac{1}{n^k} \sum_{d \in \mathcal{D}_k(n)} |A_d| = 0.$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{n^k} \sum_{d \in \mathcal{D}_k(n)} |B_d| = 0.$$

The case of $k = 1$ is treated in Lemma 3.3. Let us assume this is true up to an integer k .

Let $d \in \mathcal{D}_{k+1}(n)$. For clarity in the fomulas, let us write $d = (d_1, \dots, d_{k-1}, j, l)$ and $d' = (d_1, \dots, d_{k-1}, j) \in \mathcal{D}_k(n)$.

$$\Pi_d = \Pi_{d'} \tilde{I}_{a_{j+1}} \cdots \tilde{I}_{a_{l-1}} \tilde{\beta}_{a_l}$$

Now compute:

$$\tilde{I}_{a_{j+1}} \cdots \tilde{I}_{a_{l-1}} \tilde{\beta}_{a_l} = \begin{pmatrix} \frac{1}{2} - b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} & 0 \\ -\frac{b_l}{2^{l-j}} & 0 \end{pmatrix}$$

and by induction hypothesis,

$$\Pi_{d'} = \begin{pmatrix} \frac{1}{2^k} + A_{d'} & 0 \\ B_{d'} & 0 \end{pmatrix}$$

Thus

$$\Pi_d = \begin{pmatrix} \frac{1}{2^{k+1}} - \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} + \frac{A_{d'}}{2} - A_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} & 0 \\ \frac{1}{2} B_{d'} - B_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} & 0 \end{pmatrix}$$

and we have to prove the following:

Claim:

$$\frac{1}{n^{k+1}} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| -\frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} + \frac{A_{d'}}{2} - A_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof of the claim:

- First, we have:

$$\begin{aligned} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| &= \sum_{j=k}^{n-k} \sum_{0 \leq d_1 < \dots < d_{k-1} < j} \sum_{l=j+1}^n \left| \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \\ &\leq n^{k-1} \sum_{j=k}^{n-k} \sum_{l=j+1}^n \left| \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \end{aligned}$$

and notice that $\sum_{l=j+1}^n \left| b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \leq C_{2,n}$ so, for n big enough, according to Lemma 3.3:

$$\begin{aligned} &n^{k-1} \sum_{j=k}^{n-k} \sum_{l=j+1}^n \left| \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \\ &\leq \frac{n^{k-1}}{2^k} \sum_{j=k}^{n-k} n^{\frac{1}{2} + \varepsilon} \\ &\leq \frac{n^{k+\frac{1}{2} + \varepsilon}}{2^k} \end{aligned}$$

- Also,

$$\sum_{d \in \mathcal{D}_{k+1}(n)} \left| \frac{A_{d'}}{2} \right| = \sum_{d' \in \mathcal{D}_k(n)} \sum_{l=k}^n \left| \frac{A_{d'}}{2} \right| \leq n \sum_{d' \in \mathcal{D}_k(n)} \left| \frac{A_{d'}}{2} \right|$$

and, by induction,

$$\lim_{n \rightarrow +\infty} \frac{1}{n^k} \sum_{d' \in \mathcal{D}_k(n)} \left| \frac{A_{d'}}{2} \right| = 0$$

hence

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{k+1}} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| \frac{A_{d'}}{2} \right| = 0.$$

- Finally:

$$\begin{aligned} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| A_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| &= \sum_{j=k}^{n-k} \sum_{0 \leq d_1 < \dots < d_{k-1} < j} \sum_{l=j+1}^n \left| A_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \\ &\leq \sum_{j=k}^{n-k} \sum_{0 \leq d_1 < \dots < d_{k-1} < j} |A_{d'}| \sum_{l=j+1}^n \left| b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \end{aligned}$$

and, as in the study of the first term, using Lemma 3.3, for n big enough:

$$\begin{aligned}
& \sum_{j=k}^{n-k} \sum_{0 \leq d_1 < \dots < d_{k-1} < j} |A_{d'}| \sum_{l=j+1}^n \left| b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \\
& \leq \sum_{j=k}^{n-k} \sum_{0 \leq d_1 < \dots < d_{k-1} < j} |A_{d'}| n^{\frac{1}{2} + \varepsilon} \\
& \leq n^{\frac{1}{2} + \varepsilon} \sum_{d' \in \mathcal{D}_k(n)} |A_{d'}| \\
& \leq n^{k + \frac{1}{2} + \varepsilon}
\end{aligned}$$

by induction hypothesis.

In the end,

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{k+1}} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| \frac{1}{2^k} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} + \frac{A_{d'}}{2} - A_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| = 0.$$

And the same goes for proving that:

$$\frac{1}{n^{k+1}} \sum_{d \in \mathcal{D}_{k+1}(n)} \left| \frac{1}{2} B_{d'} - B_{d'} b_l \sum_{i=j+1}^{l-1} \frac{b_i}{2^{l-i}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Hence,

$$\begin{aligned}
(1 \ 0) \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \sum_{d \in \mathcal{D}_k(n)} \Pi_d \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
&= \sum_{d \in \mathcal{D}_k(n)} \frac{1}{2^k} + A_d - \sum_{d \in \mathcal{D}_k(n)} B_d \\
&= \sum_{d \in \mathcal{D}_k(n)} \frac{1}{2^k} + \sum_{d \in \mathcal{D}_k(n)} A_d - \sum_{d \in \mathcal{D}_k(n)} B_d \\
&= \binom{n}{k} \frac{1}{2^k} + \sum_{d \in \mathcal{D}_k(n)} A_d - \sum_{d \in \mathcal{D}_k(n)} B_d \\
&= \frac{n!}{k!(n-k)!2^k} + \sum_{d \in \mathcal{D}_k(n)} A_d - \sum_{d \in \mathcal{D}_k(n)} B_d
\end{aligned}$$

and since

$$\lim_{n \rightarrow +\infty} \frac{1}{n^k} \sum_{d \in \mathcal{D}_k(n)} |A_d| = 0$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{n^k} \sum_{d \in \mathcal{D}_k(n)} |B_d| = 0,$$

we have that:

$$(1 \ 0) \sum_{M \in \mathcal{F}_{(\alpha^0, \beta^k)}^{(n)}} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underset{n \rightarrow +\infty}{=} \frac{n^k}{2^k k!} + o(n^k),$$

which yields:

$$\lim_{n \rightarrow +\infty} \frac{2^k m_{2k}(a_X(n))}{n^k} = \frac{(2k)!}{2^k k!}.$$

□

Hence the moments of the probability measure $\tilde{\mu}_{a_X(n)}$ converge towards the moments of the normal law $\mathcal{N}(0, 1)$, which prove the Central Limit Theorem 1.9 by [12].

References

- [1] Jean Bésineau. Indépendance statistique d'ensembles liés à la fonction “somme des chiffres”. In *Séminaire Delange-Pisot-Poitou, 13e année (1971/72), Théorie des nombres, Fasc. 2, Exp. No. 23*, page 8. Secrétariat Mathématique, Paris, 1973.
- [2] Julien Cassaigne, Christian Mauduit, and András Sárközy. On finite pseudorandom binary sequences. VII. The measures of pseudorandomness. *Acta Arith.*, 103(2):97–118, 2002.
- [3] Thomas W. Cusick, Yuan Li, and Pantelimon Stănică. On a combinatorial conjecture. *Integers*, 11:A17, 17, 2011.
- [4] Hubert Delange. Sur les fonctions q -additives ou q -multiplicatives. *Acta Arith.*, 21:285–298. (errata insert), 1972.
- [5] Hubert Delange. Sur la fonction sommatoire de la fonction “somme des chiffres”. *Enseignement Math. (2)*, 21(1):31–47, 1975.
- [6] Michael Drmota. The joint distribution of q -additive functions. *Acta Arith.*, 100(1):17–39, 2001.
- [7] Michael Drmota, Manuel Kauers, and Lukas Spiegelhofer. On a Conjecture of Cusick Concerning the Sum of Digits of n and $n + t$. *SIAM J. Discrete Math.*, 30(2):621–649, 2016.
- [8] Michael Drmota, Christian Mauduit, and Joël Rivat. Primes with an average sum of digits. *Compos. Math.*, 145(2):271–292, 2009.
- [9] Jordan Emme and Alexander Prikhodko. On the asymptotic behaviour of the correlation measure of sum-of-digits function in base 2. *Preprint*, (arXiv:1504.01701), 2015.
- [10] Michael Keane. Generalized Morse sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 10:335–353, 1968.
- [11] Dong-Hyun Kim. On the joint distribution of q -additive functions in residue classes. *J. Number Theory*, 74(2):307–336, 1999.
- [12] A.A. Markov. *Démonstration du second théorème-limite du calcul des probabilités par la méthode des moments*. Saint-Petersbourg, 1913.
- [13] Johannes F. Morgenbesser and Lukas Spiegelhofer. A reverse order property of correlation measures of the sum-of-digits function. *Integers*, 12:Paper No. A47, 5, 2012.